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### Version of attached file:

Published Version

### Peer-review status of attached file:

Peer-reviewed

### Citation for published item:

Donos, Aristomenis and Gauntlett, Jerome P. (2015) 'Navier-Stokes equations on black hole horizons and DC thermoelectric conductivity.', *Physical review D.*, 92 (12). 121901(R).

### Further information on publisher's website:

<http://dx.doi.org/10.1103/PhysRevD.92.121901>

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# Navier-Stokes equations on black hole horizons and DC thermoelectric conductivity

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(Received 29 June 2015; revised manuscript received 13 August 2015; published 14 December 2015)

Within the context of the AdS/CFT correspondence, we show that the DC thermoelectric conductivity can be obtained by solving the linearized, time-independent, and forced Navier-Stokes equations on the black hole horizon for an incompressible and charged fluid.

DOI: [10.1103/PhysRevD.92.121901](https://doi.org/10.1103/PhysRevD.92.121901)

PACS numbers: 11.25.Tq, 04.50.Gh, 04.70.Bw

## I. INTRODUCTION

A striking feature of the AdS/CFT correspondence is that some fundamental properties of the dual conformal field theory (CFT) are captured by the geometry of the black hole horizon. The temperature of the CFT is equal to the Hawking temperature of the black hole which is determined by the surface gravity of the black hole. Similarly, the entropy of the CFT is the Bekenstein-Hawking entropy which is given by one quarter of the area of the black hole event horizon. Although not universal [1], it is also interesting that for a subclass of holographic black holes the shear viscosity,  $\eta$ , is also captured by the area of the black hole horizon via  $\eta = s/4\pi$ , where  $s$  is the entropy density [2,3].

Here we will argue that, remarkably, the DC thermal conductivity and, more generally, the thermoelectric conductivity of the dual field theory, are universally captured by physics at the black hole horizon. Specifically, one needs to solve linearized, time-independent and forced Navier-Stokes equations for an incompressible charged fluid on the curved horizon.

The thermal conductivity, a property relevant for all dual field theories, determines the heat current,  $\bar{Q}^i$ , that is produced after applying a temperature gradient,  $\zeta^i = -\partial_i T/T$ , at the level of linear response. If the dual field theory has a global  $U(1)$  symmetry, then with the additional application of an electric field  $E^i$ , the heat current and electric current,  $\bar{J}^i$ , that are produced define the thermoelectric conductivities via

$$\begin{pmatrix} \bar{J}^i \\ \bar{Q}^i \end{pmatrix} = \begin{pmatrix} \sigma^{ij} & T\alpha^{ij} \\ T\bar{\alpha}^{ij} & T\bar{\kappa}^{ij} \end{pmatrix} \begin{pmatrix} E_j \\ \zeta_j \end{pmatrix}. \quad (1)$$

It is important to emphasize that  $\bar{Q}^i$ ,  $\bar{J}^i$  are the total current fluxes, defined later. In seeking applications of the AdS/CFT correspondence to real materials, the DC conductivities are important observables to study. Our results can be used to determine whether or not the dual field theory is a conductor or an insulator and, at a more refined level, the temperature dependence of the conductivities, including the appearance of any scaling laws.

From a theoretical point of view, the DC conductivities are somewhat subtle to study, however, since they are generically infinite unless there is some mechanism for momentum to dissipate. A natural framework to study momentum dissipation is provided by “holographic lattices” [4], namely, black hole solutions with asymptotic behavior at the AdS boundary that are associated with adding sources to the dual CFT which break spatial translations. There has been much interest in these black holes since they can realize metal-insulator transitions [5,6] as well as novel incoherent metals [5,7,8].

The holographic lattices depend on the holographic radial direction as well as the spatial directions and, hence, constructing them generically involves solving PDEs. As a result, most examples that have been studied are one-dimensional lattices, which break translation invariance in just one of the spatial directions. An important exception is provided by  $Q$ -lattices [6] (and similar constructions [9]) whose matter content can be used to break translational invariance periodically in all spatial dimensions while the metric remains translationally invariant.

A method for calculating the DC thermoelectric conductivity for  $Q$ -lattices and one-dimensional lattices was presented in [7,10,11]. For these lattices, the final result was expressed explicitly in terms of the black hole solution at the horizon. This substantially extended a similar result found for the DC electric conductivity at zero charge density and with no momentum dissipation [12]. Here we will show that for generic lattices, breaking translations in all spatial directions, one cannot obtain such an explicit formula for the DC conductivity. However, the DC conductivity can always be obtained by solving Navier-Stokes equations on the black hole horizon. The earlier results can now be viewed as special cases in which the fluid equations can be explicitly solved.

An early connection between gravity and fluids is the membrane paradigm [13]. More recently, in the holographic fluid-gravity correspondence [14], approximate solutions to the gravity equations are obtained by solving relativistic hydrodynamic equations for the boundary theory via a systematic derivative expansion. The Navier-Stokes equations arise after taking a scaling limit

and, hence, they too can be captured in a dual gravitational description [15,16]. These connections are applicable in a hydrodynamic limit. On the other hand, in [17] it was shown how solutions of Navier-Stokes equations on hypersurfaces in Minkowski space give rise to solutions of Einstein's equations. Here, by contrast, solutions of Navier-Stokes equations on the black hole horizon lead to exact transport quantities, given by specific two-point correlators, in the deformed dual CFT. In obtaining these results, we do not take a hydrodynamic limit of the dual field theory (or any other limit), and our results apply to arbitrary horizon geometries that arise as solutions to the equations of motion. We expect that the time-dependent and nonlinear generalization of the fluid equations that we obtain will also play a role in studying holographic lattices, for example, in a suitable hydrodynamic limit (e.g. see [18,19]). We emphasize, though, that our results here already show that independently of the strength of translation breaking effects and the temperature, a hydrodynamic description of DC transport is always possible in terms of a specific fluid living on the black hole horizon.

For simplicity, we focus on holographic lattices of  $D = 4$  Einstein-Maxwell theory; the main results extend very simply to  $D \geq 4$ , as well as the inclusion of other matter fields. More details appear in [20].

## II. BACKGROUND BLACK HOLES

We consider the  $D = 4$  bulk action [21]:

$$S = \int d^4x \sqrt{-g} \left( R + 6 - \frac{1}{4} F^2 \right). \quad (2)$$

The unit radius  $\text{AdS}_4$  vacuum solution is dual to a  $d = 3$  CFT with a global  $U(1)$  symmetry. We will focus on the class of electrically charged, static black holes given by

$$ds^2 = -UGdt^2 + \frac{F}{U} dr^2 + ds^2(\Sigma_2), \quad A = a_t dt, \quad (3)$$

where  $ds^2(\Sigma_2) \equiv g_{ij}(r, x) dx^i dx^j$  is a metric on a two-dimensional manifold,  $\Sigma_2$ , at fixed  $r$ . Also,  $U = U(r)$ , while  $G$ ,  $F$ , and  $a_t$  are all functions of  $(r, x^i)$ .

At the  $\text{AdS}_4$  boundary, as  $r \rightarrow \infty$ , we have

$$U \rightarrow r^2, \quad F \rightarrow 1, \quad G \rightarrow \bar{G}(x), \\ g_{ij}(r, x) \rightarrow r^2 \bar{g}_{ij}(x), \quad a_t(r, x) \rightarrow \mu(x). \quad (4)$$

The spatial dependence of the boundary metric, given by  $\bar{G}(x)$ ,  $\bar{g}_{ij}(x)$ , corresponds to a source for the stress tensor of the dual CFT. Similarly,  $\mu(x)$  is a spatially dependent chemical potential for the global  $U(1)$  symmetry. An interesting subclass of solutions is associated with adding spatially periodic sources to a CFT in flat space. In this

case, the functions are all periodic in the spatial coordinates  $x^i$ , and we can, in effect, take  $\Sigma_2$  to be a torus.

The black hole horizon is assumed to be located at  $r = 0$ . By considering the Kruskal coordinate  $v = t + \frac{\ln r}{4\pi T} + \dots$ , we deduce that the near-horizon expansions are given by

$$U(r) = r(4\pi T + U^{(1)}r + \dots), \\ a_t(r, x) = r(a_t^{(0)}(x)G^{(0)}(x) + a_t^{(1)}(x)r + \dots), \\ G(r, x) = G^{(0)}(x) + G^{(1)}(x)r + \dots, \\ F(r, x) = G^{(0)}(x) + F^{(1)}(x)r + \dots, \\ g_{ij}(r, x) = g_{ij}^{(0)}(x) + g_{ij}^{(1)}(x)r + \dots, \quad (5)$$

where the factor of  $G^{(0)}$  in the leading term of  $a_t(r, x)$  has been added so that electric charge density at the horizon is simply  $\sqrt{-g}F^{tr}|_H = \sqrt{-g_0}a_t^{(0)}$ .

## III. PERTURBING THE BLACK HOLES

We will consider a perturbation that provides sources  $E$ ,  $\zeta$  for the electric and heat currents, respectively, that are linear in  $t$ . Specifically, generalizing [7,10,11], we study

$$\delta(ds^2) = \delta g_{\mu\nu} dx^\mu dx^\nu - 2tGU\zeta_i dt dx^i, \\ \delta A = \delta a_\mu dx^\mu - tE_i dx^i + ta_i \zeta_i dx^i, \quad (6)$$

with  $\delta g_{\mu\nu}$ ,  $\delta a_\mu$  functions of  $(r, x^i)$ , while  $E_i = E_i(x)$ ,  $\zeta_i = \zeta_i(x)$  are one-forms on  $\Sigma_2$ , and we demand that

$$d(E_i dx^i) = d(\zeta_i dx^i) = 0. \quad (7)$$

This perturbation solves the time dependence of the equations of motion at linear order.

At the  $\text{AdS}_4$  boundary, we demand that the falloff of  $\delta g_{\mu\nu}$ ,  $\delta a_\mu$  is such that the only sources are parametrized by  $E$ ,  $\zeta$ . At the black hole horizon, as  $r \rightarrow 0$ , regularity implies that we must have

$$\delta g_{tt} = U(\delta g_{tt}^{(0)}(x) + \mathcal{O}(r)), \quad \delta g_{rr} = \frac{1}{U}(\delta g_{rr}^{(0)}(x) + \mathcal{O}(r)), \\ \delta g_{ij} = \delta g_{ij}^{(0)}(x) + \mathcal{O}(r), \quad \delta g_{tr} = \delta g_{tr}^{(0)}(x) + \mathcal{O}(r), \\ \delta g_{ti} = \delta g_{ti}^{(0)}(x) - \zeta_i GU \frac{\ln r}{4\pi T} + \mathcal{O}(r), \\ \delta g_{ri} = \frac{1}{U}(\delta g_{ti}^{(0)}(x) + \mathcal{O}(r)), \\ \delta a_t = \delta a_t^{(0)}(x) + \mathcal{O}(r), \quad \delta a_r = \frac{1}{U}(\delta a_t^{(0)}(x) + \mathcal{O}(r)), \\ \delta a_i = \frac{\ln r}{4\pi T}(-E_i + a_t \zeta_i) + \mathcal{O}(r), \quad (8)$$

with  $\delta g_{tt}^{(0)} + \delta g_{rr}^{(0)} - 2\delta g_{tr}^{(0)} = 0$ . Note that the logarithm terms combine with the terms linear in time in (6).

### A. Electric and heat currents

We define the bulk electric current density as

$$J^i = \sqrt{-g}F^{ir}. \quad (9)$$

At the  $\text{AdS}_4$  boundary, we find that  $J^i|_\infty$  is the electric current density of the dual field theory. The gauge equations of motion,  $\nabla_\mu F^{\mu\nu} = 0$ , imply

$$\partial_t J^i = 0, \quad \partial_r J^i = \partial_j(\sqrt{-g}F^{ji}), \quad (10)$$

and another equation which will not play a further role.

For the heat currents, we want to identify equations of motion involving the metric perturbation that have a similar structure to the gauge equations of motion. First, consider a vector  $k$  which satisfies

$$\nabla_\mu k^\mu = 0, \quad \nabla_\mu \nabla^{(\mu} k^{\nu)} = \alpha k^\nu, \quad (11)$$

for some function  $\alpha$ , which would vanish if  $k$  is a Killing vector. We also write  $\varphi = k^\mu A_\mu$  and  $k^\mu F_{\mu\nu} = \partial_\nu \theta + s_\nu$ , with  $s$  a one-form and  $\theta$  a globally defined function. In the special case that the Lie derivative of  $F$  with respect to  $k$  vanishes, we have  $\partial_{[\mu} s_{\nu]} = 0$ . We now define the two-form  $G$ :

$$G^{\mu\nu} = -2\nabla^{[\mu} k^{\nu]} - k^{[\mu} F^{\nu]\sigma} A_\sigma - \frac{1}{2}(\varphi - \theta)F^{\mu\nu}. \quad (12)$$

The equations of motion then imply that

$$\nabla_\mu G^{\mu\nu} = (\alpha - 6)k^\nu + \frac{1}{2}F^{\nu\rho}s_\rho - \frac{1}{2}(\mathcal{L}_k F)^{\nu\rho}A_\rho, \quad (13)$$

where  $\mathcal{L}_k$  is the Lie derivative with respect to  $k$ . In our case,  $k = \partial_t$ , and at linearized order,  $\varphi = a_t + \delta a_t$ ,  $\theta = -a_t - \delta a_t$ ,  $s = -E_i dx^i + a_t \zeta_i dx^i$ , and  $\alpha \neq 0$  when  $\zeta \neq 0$ .

We can now define the following bulk current density:

$$Q^i = \sqrt{-g}G^{ir}. \quad (14)$$

From (13) we deduce, in particular, that

$$\partial_t Q^i = 0, \quad \partial_r Q^i = -\partial_j(2\sqrt{-g}G^{ji}). \quad (15)$$

By calculating the holographic stress tensor,  $t^{\mu\nu}$ , we find

$$\bar{G}^{3/2}\sqrt{\bar{g}}t^{ti} - \mu J^i|_\infty = Q^i|_\infty - t\bar{G}^{3/2}\sqrt{\bar{g}}t^{ij}\zeta_j, \quad (16)$$

and we conclude that  $Q^i|_\infty$  is the time-independent part of the heat current density [22].

### B. Navier-Stokes on the horizon

The next step in our analysis is to examine the equations of motion for the perturbed black holes in the context of a Hamiltonian decomposition with respect to the radial

direction. As is well known, the Hamiltonian is simply a sum of constraints. We want to evaluate the constraints as an expansion in the radius at the black hole horizon. The details of this calculation are technically involved and will be fully described in [20]. The final results, however, are simple to explain. We find that the Gauss law constraint implies that  $\partial_i J^i_{(0)} = 0$ , where  $J^i_{(0)} \equiv J^i|_H$ . Furthermore, the  $t$  components of the momentum constraint,  $H_t = 0$ , as well as the Hamiltonian constraint,  $H = 0$ , each separately imply that  $\partial_i Q^i_{(0)} = 0$  with  $Q^i_{(0)} \equiv Q^i|_H$ . Finally, the  $i$  components of the momentum constraint,  $H_i = 0$ , give additional equations which, when combined with the others, gives the linearized Navier-Stokes equations on the black hole horizon, presented below.

To summarize, evaluating the constraints at the horizon leads to a closed system of equations for a subset of the perturbation which must be satisfied at the black hole horizon. The black hole horizon is as in (3)–(5). The perturbation at the horizon is given as in (6)–(8), and it is illuminating to introduce the following notation:

$$v_i \equiv -\delta g_{it}^{(0)}, \quad w \equiv \delta a_t^{(0)},$$

$$p \equiv -4\pi T \frac{\delta g_{tt}^{(0)}}{G^{(0)}} - g_{(0)}^{ij} \nabla_j \ln G^{(0)} \delta g_{it}^{(0)}. \quad (17)$$

The current densities at the horizon can be written as

$$J^i_{(0)} = \rho_H v^i + \sigma_H^{ij}(\partial_j w + E_j),$$

$$Q^i_{(0)} = T s_H v^i, \quad (18)$$

where we define the horizon quantities as follows:

$$\rho_H = \sqrt{g_{(0)}}a_t^{(0)}, \quad s_H = 4\pi\sqrt{g_{(0)}},$$

$$\sigma_H^{ij} = \sqrt{g_{(0)}}g_{(0)}^{ij}, \quad \eta_H = \frac{s_H}{4\pi}. \quad (19)$$

The four unknowns in (17) satisfy the following system of four linear partial differential equations:

$$\nabla_i v^i = 0, \quad (20)$$

$$\nabla^2 w + v^i \nabla_i(a_t^{(0)}) = -\nabla_i E^i, \quad (21)$$

$$\eta_H[-2\nabla^i \nabla_{(i} v_{j)} + \nabla_j p] = T s_H \zeta_j + \rho_H(E_j + \partial_j w), \quad (22)$$

where the covariant derivatives are with respect to the metric on the black hole horizon  $g_{ij}^{(0)}$ , and all indices are being raised and lowered with this metric. The first two equations are simply  $\partial_i Q^i_{(0)} = \partial_i J^i_{(0)} = 0$ . Note that in (22) we can also write  $2\nabla^i \nabla_{(i} v_{j)} = \nabla^2 v_j + R_{ji} v^i$ .

Remarkably, we have obtained the time-independent, linearized Navier-Stokes equations for a forced, incompressible, charged fluid on the curved horizon. Such equations are also called Stokes equations. The fluid velocity is  $v_i$ , the effective pressure is  $p$ , and  $w$  is a scalar

potential. The forcing terms are given by the one-forms  $4\pi T\zeta$  and  $E$ . In (18) and (22), we see that  $\rho_H$ ,  $s_H$ ,  $\sigma_H^{ij}$ , and  $\eta_H$  can be viewed as coefficients in the constitutive relations for the horizon fluid:  $\rho_H$  and  $s_H$  are the charge density and the entropy density, while  $\sigma_H^{ij}$  and  $\eta_H$  are transport coefficients associated with the electric conductivity and the shear viscosity of the horizon fluid, respectively. We stress that  $\eta_H$ ,  $\sigma_H^{ij}$  are *not*, in general, the shear viscosity and DC electric conductivity of the deformed dual CFT [23]. It is also worth noting that possible thermoelectric transport coefficients for the black hole horizon fluid  $\alpha_H$ ,  $\bar{\alpha}_H$ , and  $\bar{\kappa}_H$  are all absent in the expressions (18).

We now establish a number of interesting properties of this set of equations. First, we multiply (22) by  $v_j$  and then integrate over the horizon, leading to

$$\int d^2x \sqrt{g_0} [2\nabla^{(i} v^{j)} \nabla_{(i} v_{j)} + (\nabla w + E)^2] = \int d^2x (Q_{(0)}^i \zeta_i + J_{(0)}^i E_i). \quad (23)$$

In the case of noncompact horizons, we have assumed that possible boundary terms vanish. Observe that the left-hand side is a manifestly positive quantity, and this is associated with the thermoelectric conductivities being a positive semidefinite matrix.

Second, we consider the issue of uniqueness for (20)–(22). If we have two solutions, then the difference of the functions will satisfy the same equations but with vanishing forcing terms,  $\zeta = E = 0$ . Denoting the difference by  $(v_i, w, p)$ , we immediately conclude from (23) that

$$\nabla^{(i} v^{j)} = 0, \quad \nabla_i w = 0. \quad (24)$$

We also have  $v^i \partial_i a_t^{(0)} = 0$  from (21) and  $\nabla p = 0$  from (22). We conclude that the solution space is unique up to Killing vectors of the horizon metric, with  $p$ ,  $w$  constant and  $\delta g_{rt}$  fixed by (17). This results agrees with the intuition that one should be able to boost along the orbits of Killing vectors to obtain a solution with momentum.

Third, we observe that when  $(E, \zeta)$  are exact forms,  $(E, \zeta) = (de, dz)$  with  $e$ ,  $z$  globally defined functions on  $\Sigma_2$ , we can solve Eqs. (20)–(22) by taking  $w = -e$  and  $p = 4\pi Tz$ , plus possible constants, and  $v^i = 0$ . We observe that this solution gives zero contribution to the current densities (18) at the horizon. This solution gives no contribution to the DC thermoelectric conductivity, which we discuss below. Thus, the DC conductivity is determined by the harmonic part of  $E$  and  $\zeta$ .

### C. The thermoelectric DC conductivity

We have shown that the electric and heat currents at the horizon, given in (18), can be expressed in terms of the sources  $E$ ,  $\zeta$  after solving the Stokes equations (20)–(22). To obtain DC conductivities of the field theory, we need to relate

the currents at the black hole horizon to the currents at the AdS boundary. In some cases, these are the same. In general, however, the currents depend on the radial coordinate  $r$  and one needs to integrate (10), (15). In general, however, we can always define total current fluxes which are independent of  $r$  and hence obtain associated DC conductivities.

As a concrete example, assume [24] we have a periodic holographic lattice on  $\mathbb{R}^{1,2}$ . That is, the lattice deformations  $\tilde{G}(x)$ ,  $\tilde{g}_{ij}(x)$  and  $\mu(x)$  in (4) are periodic functions of the spatial coordinates  $(x^1, x^2) \sim (x^1 + L_1, x^2 + L_2)$ . Defining the total electric current flux densities through the  $x_2$  plane or the  $x_1$  plane, respectively,

$$\bar{J}^1 \equiv \frac{1}{L_2} \int J^1 dx^2, \quad \bar{J}^2 \equiv \frac{1}{L_1} \int J^2 dx^1, \quad (25)$$

and defining  $\bar{Q}^i$  in a similar way, we can immediately deduce from (10) and (15) that  $\partial_r \bar{J}^i = \partial_r \bar{Q}^i = 0$ , which is just Stokes's theorem in the bulk. These current fluxes at the AdS boundary are, thus, given by their values at the black hole horizon which, in turn, are fixed by  $E$ ,  $\zeta$  after solving (20)–(22) in order to obtain  $v^i$  and  $w$  and then using (18). These data then give the DC thermoelectric conductivities via (1).

## IV. EXAMPLES

For the special case that the lattice depends on only one of the spatial coordinates, we can explicitly solve the Stokes equations and recover the results of [11]. We will present the details of this calculation in [20]. Here we will consider the case of a perturbative and periodic lattice associated with coherent metals with Drude peaks [25]. Specifically, we consider perturbative solutions about the AdS-RN black brane with a flat horizon. If  $\lambda$  is the expansion parameter, at the horizon we assume

$$G^{(0)} = f_{(0)} + \lambda f_{(1)} + \dots, \quad g_{ij}^{(0)} = g\delta_{ij} + \lambda h_{ij}^{(1)} + \dots, \quad a_t^{(0)} = a + \lambda a_{(1)} + \dots, \quad (26)$$

with  $f_{(0)}$ ,  $g$ , and  $a$  constants, and the remaining functions are periodic on the torus  $\Sigma_2$ . For the Ricci tensor, we have  $R_{(0)ij} = \lambda R_{ij}^{(1)} + \lambda^2 R_{ij}^{(2)} + \dots$  and similarly for the Christoffel symbols.

We can solve (20)–(22) perturbatively, using the expansion  $v = \frac{1}{\lambda^2} v_{(0)} + \frac{1}{\lambda} v_{(1)} + \dots$ ,  $w = \frac{1}{\lambda} w_{(1)} + w_{(2)} + \dots$ , and  $p = \frac{1}{\lambda} p_{(1)} + p_{(2)} + \dots$ . At leading order we find that  $\partial_i v_{(0)}^i = 0$ ,  $\square v_{(0)}^i = 0$ , implying that  $v_{(0)}^i$  are constant on the torus. At next order, we have  $v_{(1)}^i = N_{(1)j}^i v_{(0)}^j$  with

$$N_{(1)j}^i = -\square^{-1} (\partial^k (\Gamma^{(1)})_{kj}^i + R^{(1)i}{}_j - \partial^i (\square^{-1} \partial_j R^{(1)})). \quad (27)$$

Note that the function  $\square^{-1} f$  is defined up to a constant on a torus. Such constants are fixed at third order in the



expansion, and they do not affect the DC result at leading order.

We next integrate equation (22) on the black hole horizon, discarding any boundary terms in the non-compact case, and keep the  $\lambda^0$  part. After defining  $\Sigma_j \equiv (4\pi T \zeta_j + a E_j)$ , now with  $\zeta_i, E_i$  constant on the torus, we deduce that, at leading order in  $\lambda$ , we have

$$v^i \approx (L^{-1})^{ij} \Sigma_j, \quad J_{(0)}^i \approx \rho_H v^i, \quad Q_{(0)}^i \approx T s v^i. \quad (28)$$

Here  $\rho_H = g^2 a$  is the charge density at the horizon,  $s = 4\pi g^2$  is the entropy density, and the constant matrix  $L$  is given by

$$L_{ji} = \lambda^2 g^{-1} \int_H \left( \frac{g^{-1}}{2} \partial_j h_{kl}^{(1)} \partial_i h^{(1)kl} + \partial_j h_{kl}^{(1)} \partial^k N_{(1)i}^l + \frac{1}{2} h^{(1)} \partial_j (\square^{-1} \partial_i R^{(1)}) + g^2 a_{(1)} \partial_j (\square^{-1} \partial_i a_{(1)}) \right), \quad (29)$$

where  $N_{(1)}$  is given by (27) and  $h^{(1)} = h_{ij}^{(1)} \delta^{ij}$ .

Using (25) we can write  $\bar{J}^i \approx \rho v^i$ ,  $\bar{Q}^i \approx T s v^i$ , where  $\rho = (L_1 L_2)^{-1} \int dx^1 dx^2 \sqrt{-g} F^{tr}|_\infty = \rho_H$  is the total averaged charge density. Hence, from (1) the DC conductivities are given by  $\bar{\kappa}^{ij} = (L^{-1})^{ij} 4\pi s T$ ,  $\alpha = \bar{\alpha} = (L^{-1})^{ij} 4\pi \rho$ ,  $\sigma = (L^{-1})^{ij} 4\pi \rho^2 / s$ .

It is interesting to observe, for this general class of holographic lattices, that at leading order  $\bar{\kappa}^{ij}(\sigma T)_{jk}^{-1} = s^2 / \rho^2 \delta_k^i$ , corresponding to a kind of Wiedemann-Franz law. Also the thermal conductivity when  $J = 0$ ,  $\kappa \equiv \bar{\kappa} - T \bar{\alpha} \sigma^{-1} \alpha$  as well as the electric conductivity when  $Q = 0$ ,  $\sigma_{Q=0} \equiv \sigma - T \alpha \bar{\kappa}^{-1} \bar{\alpha}$  appear at order  $\lambda^0$  in the expansion. These general results complement those using other techniques in [26].

## ACKNOWLEDGMENTS

The work is supported by STFC Grant No. ST/J0003533/1, EPSRC Grant No. EP/K034456/1, and by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013), ERC Grant Agreement No. ADG 339140.

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